Lie Symmetries of (1+1)-Dimensional Cubic Schrödinger Equation with Potential

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We perform the complete group classification in the class of cubic Schrödinger equations of the form $i\psi_t + \psi_{xx} + \psi^2\psi^* + V(t,x)\psi = 0$, where V is an arbitrary complex-valued potential depending on t and x. We construct all possible inequivalent potentials for which these equations have non-trivial Lie symmetries using algebraic and compatibility methods simultaneously. Our classification essentially amends earlier works on the subject.

Nonlinear Schrödinger equations (NSchEs) have a number of applications in wave propagation in inhomogeneous media. They arise as a model of plasma phenomena, namely, of different processes in nonlinear and non-uniform dielectric medium and in other branches of physics. Schrödinger equations have been investigated by means of symmetry methods by a number of authors, see e.g. [2, 3, 4, 5, 6, 7, 8, 9, 10] and references there. In fact, group classification for Schrödinger equations was first performed by S. Lie. More precisely, his classification [1] of all the linear equations with two independent complex variables contains, in an implicit form, solution of the classification problem for the linear (1+1)-dimensional Schrödinger equations with arbitrary potentials.

In this paper we study a class of NSchEs of the form

$$i\psi_t + \psi_{xx} + \psi^2\psi^* + V\psi = 0, \tag{1}$$

where the potential $V = V(t,x)$ is an arbitrary complex-valued smooth function of the variables t and x. (Here and below subscripts of functions denote differentiation with respect to the corresponding variables.) To find a complete set of inequivalent cases of V admitting extensions of the maximal Lie invariance algebra, we combine the classical Lie approach, studying the algebra generated by all the possible Lie symmetry operators for equations from class (1) (the adjoint representation, the inequivalent one-dimensional subalgebras etc.) and investigation of compatibility of classifying equations. See [2, 3, 11, 12, 13] for precise formulation of group classification problems and more details on the used methods.

Finishing excellent series of papers [8, 9, 10] on group analysis and exact solutions of NSchEs, in [10] L. Gagnon and P. Winternitz investigated essentially more general class of variable coefficient NSchEs than (1). Unfortunately, we were not able to see a direct and simple way for deducing classifications obtained here from their results.

Theorem 1. *Any operator $Q = \xi^t\partial_t + \xi^x\partial_x + \eta\partial_\psi + \eta^*\partial_{\psi^*}$ from the maximal Lie invariance algebra $A^{\max}(V)$ of equation* (1) *with arbitrary potential V lies in the linear span of operators of the form*

$$D(\xi) = \xi\partial_t + \tfrac{1}{2}\xi_t x\partial_x + \tfrac{1}{8}\xi_{tt}x^2M - \tfrac{1}{2}\xi_t I, \qquad G(\chi) = \chi\partial_x + \tfrac{1}{2}\chi_t xM, \qquad \lambda M. \tag{2}$$

Here $\chi=\chi(t)$, $\xi=\xi(t)$ and $\lambda=\lambda(t)$ are arbitrary smooth functions of t, $M=i(\psi\partial_\psi-\psi^*\partial_{\psi^*})$, $I=\psi\partial_\psi+\psi^*\partial_{\psi^*}$. Moreover, the coefficients of Q should satisfy the classifying condition

$$i\eta_{\psi t}+\eta_{\psi xx}+\xi^tV_t+\xi^xV_x+\xi^t_tV=0. \tag{3}$$

Note 1. The linear span of operators of the form (2) is an (infinite-dimensional) Lie algebra $A^\cup$ under the usual Lie bracket of vector fields. Since for any $Q\in A^\cup$ where $(\xi^t,\xi^x)\neq(0,0)$ we can find V satisfying condition (3) then $A^\cup=\langle\bigcup_V A^{\max}(V)\rangle$. The non-zero commutation relations between the basis elements of $A^\cup$ are the following ones:

$$\begin{gathered}\big[D(\xi^1),D(\xi^2)\big]=D\big(\xi^1\xi^2_t-\xi^2\xi^1_t\big),\qquad [D(\xi),G(\chi)]=G\big(\xi\chi_t-\tfrac12\,\xi_t\chi\big),\\ [D(\xi),\lambda M]=\xi\lambda_tM,\qquad \big[G(\chi^1),G(\chi^1)\big]=\tfrac12\big(\chi^1\chi^2_t-\chi^2\chi^1_t\big)M.\end{gathered}$$

We use the notation $\mathrm{Aut}(A^\cup)$ for the automorphism group acting on $A^\cup$, which is generated by all the one-parameter groups corresponding to the adjoint representations of operators of $A^\cup$ into $A^\cup$ and two discrete transformations $\mathrm{Ad}\,I_x$ and $\mathrm{Ad}\,I_t$ included additionally. The actions of $\mathrm{Ad}\,I_x$ and $\mathrm{Ad}\,I_t$ on the basis elements of $A^\cup$ are defined by the formulas $\mathrm{Ad}\,I_x\;G(\chi)=G(-\chi)$ (the other basis operators do not change) and $\mathrm{Ad}\,I_t\;D(\xi)=D(\tilde\xi)$, $\mathrm{Ad}\,I_t\;G(\chi)=G(\tilde\chi)$, $\mathrm{Ad}\,I_t\;\lambda M=\tilde\lambda M$, where $\tilde\xi(t)=-\xi(-t)$, $\tilde\chi(t)=\chi(-t)$ and $\tilde\lambda(t)=-\lambda(-t)$.

Theorem 2. *The Lie algebra of the kernel of maximal Lie invariance groups of equations from class* (1) *is $A^{\ker}=\langle M\rangle$.*

Theorem 3. *The Lie algebra $A^{\rm equiv}$ of the equivalence group $G^{\rm equiv}$ of the class* (1) *is generated by the operators*

$$\begin{gathered}D'(\xi)=D(\xi)+\tfrac18\xi_{ttt}x^2(\partial_V+\partial_{V^*})+\tfrac{i}{2}\xi_{tt}(\partial_V-\partial_{V^*})-\xi_t(V\partial_V+V^*\partial_{V^*}),\\ G'(\chi)=G(\chi)+\tfrac12\chi_{tt}x(\partial_V+\partial_{V^*}),\qquad M'(\lambda)=\lambda M+\lambda_t(\partial_V+\partial_{V^*}).\end{gathered}$$

Therefore, $A^{\rm equiv}\simeq A^\cup$, and the isomorphism is determined by means of prolongation of operators from $A^\cup$ to the space (V,V^*).

Theorem 4. *The equivalence group $G^{\rm equiv}$ of the class* (1) *is generated by the family of continuous transformations*

$$\begin{gathered}\tilde t=T,\quad \tilde x=x\varepsilon\sqrt{T_t}+X,\quad \tilde\psi=\psi\frac{1}{\sqrt{T_t}}\exp\left(\frac{i}{8}\frac{T_{tt}}{T_t}x^2+\frac{i}{2}\frac{X_t}{\sqrt{T_t}}x+i\Psi\right),\\ \tilde V=\frac{1}{T_t}\left(V+\frac18\left(\frac{T_{tt}}{T_t}\right)_t x^2+\frac12\left(\frac{X_t}{\sqrt{T_t}}\right)_t x+\frac{i}{4}\frac{T_{tt}}{T_t}-\left(\frac14\frac{T_{tt}}{T_t}x+\frac12\frac{X_t}{\sqrt{T_t}}\right)^2+\Psi_t\right),\end{gathered}\tag{4}$$

and two discrete transformations: the space reflection I_x ($\tilde t=t$, $\tilde x=-x$, $\tilde\psi=\psi$, $\tilde V=V$) and the Wigner time reflection I_t ($\tilde t=-t$, $\tilde x=x$, $\tilde\psi=\psi^*$, $\tilde V=V^*$). Here T, X and Ψ are arbitrary smooth functions of t, $T_t>0$.

Corollary 1. 1. $G^{\rm equiv}\simeq\mathrm{Aut}\,A^\cup$. 2. *Let A^1 and A^2 be the maximal Lie invariance algebras of equations from class* (1) *for some potentials, and $\mathcal V^i=\{V\,|\,A^{\max}(V)=A^i\}$, $i=1,2$. Then $\mathcal V^1\sim\mathcal V^2 \bmod G^{\rm equiv}$ iff $A^1\sim A^2 \bmod \mathrm{Aut}\,A^\cup$.*

Lemma 1. *A complete list of $\mathrm{Aut}\,A^\cup$-inequivalent one-dimensional subalgebras of $A^\cup$ is exhausted by the algebras $\langle\partial_t\rangle$, $\langle\partial_x\rangle$, $\langle tM\rangle$, $\langle M\rangle$.*

Proof. Consider any operator $Q \in A^{\cup}$, i.e. $Q = D(\xi) + G(\chi) + \lambda M$. Depending on the values of ξ, χ and λ it is equivalent under $\mathrm{Aut}\, A^{\cup}$ and multiplication by a number to one from the following operators: $D(1)$ if $\xi \neq 0$; $G(1)$ if $\xi = 0$ and $\chi \neq 0$; tM if $\xi = \chi = 0$, $\lambda_t \neq 0$; M if $\xi = \chi = \lambda_t = 0$. □

Corollary 2. *If* $A^{\max}(V) \neq A^{\ker}$ *then* $V_t V_x = 0 \bmod G^{\mathrm{equiv}}$.

Proof. Under the corollary assumption there exists an operator $Q = D(\xi) + G(\chi) + \lambda M \in A^{\max}(V)$ which do not belong to $\langle M \rangle$. Condition (3) implies $(\xi, \chi) \neq (0, 0)$. Therefore, in force of Lemma 1 $\langle Q \rangle \sim \langle \partial_t \rangle$ or $\langle \partial_x \rangle \bmod \mathrm{Aut}\, A^{\cup}$, i.e. $V_t V_x = 0 \bmod G^{\mathrm{equiv}}$. □

Theorem 5. *A complete set of inequivalent cases of V admitting extensions of the maximal Lie invariance algebra of equations* (1) *is exhausted by the potentials given in Table 1.*

Table 1. Results of classification. Here $W(t), \nu, \alpha, \beta \in \mathbb{R}$, $(\alpha, \beta) \neq (0, 0)$.

N	V	Conditions mod G^{equiv}	Basis of $A^{\max}$
0	$V(t,x)$		M
1	$iW(t)$		$M,\ \partial_x,\ G(t)$
2	$\dfrac{i}{2}\dfrac{t+\nu}{t^2+1}$	$\nu \geq 0$	$M,\ \partial_x,\ G(t),\ D(t^2+1)$
3	$i\nu t^{-1},\ \nu \neq 0, \frac{1}{2}$	$\nu \geq \frac{1}{4}$	$M,\ \partial_x,\ G(t),\ D(t)$
4	i		$M,\ \partial_x,\ G(t),\ \partial_t$
5	0		$M,\ \partial_x,\ G(t),\ \partial_t,\ D(t)$
6	$V(x)$		$M,\ \partial_t$
7	$(\alpha + i\beta)x^{-2}$	$\beta \geq 0$	$M,\ \partial_t,\ D(t)$

If we use Corollary 2, then to prove Theorem 5 it is sufficient to study two cases: $V_x = 0$ and $V_t = 0$. In fact, below we obtain the complete results of group classifications for both special cases and then unite them for the general case under consideration.

Lemma 2. *Let $V_x = 0$, i.e. $V = V(t)$.*

1. $A^{\ker}_{V_x=0} = \langle M, G(1), G(t) \rangle$. $A^{\mathrm{equiv}}_{V_x=0} = \langle M'(\lambda)\ \forall \lambda = \lambda(t), G'(1), G'(t), D'(1), D'(t), D'(t^2) \rangle$. $G^{\mathrm{equiv}}_{V_x=0}$ *is generated by I_t, I_x and the transformations of form* (4) *where $X = c_1 t + c_0$, $T = (a_1 t + a_0)/(b_1 t + b_0)$, Ψ is an arbitrary smooth function of t. a_i, b_i and c_i are arbitrary constants such that* $a_1 b_0 - b_1 a_0 > 0$.

2. $V \sim iW \bmod G^{\mathrm{equiv}}_{V_x=0}$ *where* $W = \mathrm{Im} V$. $A^{\max}(iW) \subset A^{\cup}_{\{iW\}} = A^{\ker}_{V_x=0} \not\ni \langle D(1), D(t), D(t^2) \rangle$. $A^{\ker}_{\{iW\}} = A^{\ker}_{V_x=0}$. $A^{\cup}_{\{iW\}} = \langle \bigcup_W A^{\max}(iW) \rangle$. $A^{\mathrm{equiv}}_{\{iW\}} = \langle M, G'(1), G'(t), D'(1), D'(t), D'(t^2) \rangle$. $G^{\mathrm{equiv}}_{\{iW\}} = G^{\mathrm{equiv}}_{V_x=0}\Big|_{\Psi=\mathrm{const}}$. $A^{\cup}_{\{iW\}} \simeq A^{\mathrm{equiv}}_{\{iW\}} = \mathrm{pr}_{(V,V^*)}\, A^{\cup}_{\{iW\}}$.

3. $S = \langle D(1), D(t), D(t^2) \rangle \simeq sl(2, \mathbb{R})$. *The complete list of* $\mathrm{Aut}\, A^{\cup}_{\{iW\}}$*-inequivalent proper subalgebras of S is exhausted by* $\langle D(1) \rangle$, $\langle D(t) \rangle$, $\langle D(t^2+1) \rangle$, $\langle D(1), D(t) \rangle$.

4. *Let A^1 and A^2 be the maximal Lie invariance algebras of equations from class (1) for some potentials from $\{iW(t)\}$, and $\mathcal{W}^i = \{\, W(t) \,|\, A^{\max}(iW) = A^i \}$, $i = 1, 2$. Then $\mathcal{W}^1 \sim \mathcal{W}^2$* $\bmod G^{\mathrm{equiv}}_{\{iW\}}$ *iff* $A^1 \bigcap S \sim A^2 \bigcap S \bmod \mathrm{Aut}\, S$.

5. *If $A^{\max}_{\{iW\}} \neq A^{\ker}_{V_x=0}$ the potential $iW(t)$ is $G^{\mathrm{equiv}}_{\{iW\}}$-equivalent to one from Cases 2–5 of Table 1.*

Note 2. For any W $A^{\max}(iW) \not\supset S$ (otherwise, condition (3) would imply an incompatible system for W). If $W = \mathrm{const}$ $W \in \{0,1\} \bmod G^{\mathrm{equiv}}_{\{iW\}}$ (Cases 5 and 4 correspondingly). Cases 2_ν and $2_{\tilde\nu}$ (3_ν and $3_{\tilde\nu}$ where $\nu, \tilde\nu \geq \frac{1}{4}$) are G^{equiv}-inequivalent if $\nu \neq \tilde\nu$. Since $D(t^2+1)$ cannot be contained in any two-dimensional subalgebra of S it is not possible to extend $A^{\max}$ in Case 2. There are two possibilities for extension of $A^{\max}(i\nu t^{-1})$, namely with either $D(1)$ (for $\nu = 0$, Case 5) or $D(t^2)$ (for $\nu = \frac{1}{2}$ that is equivalent to Case 5 with respect to $G^{\mathrm{equiv}}_{\{iW\}}$).

Lemma 3. *Let $V_t = 0$, i.e. $V = V(x)$.*

1. $A^{\ker}_{V_t=0} = \langle M, D(1)\rangle$. $A^{\mathrm{equiv}}_{V_t=0} = \langle M'(1), M'(t), G'(1), D'(1), D'(t)\rangle$. $G^{\mathrm{equiv}}_{V_t=0}$ *consists of I_t, I_x and the transformations of form (4) where $T_{tt} = X_t = \Psi_{tt} = 0$.*
2. *If $A^{\max}(V) \neq A^{\ker}_{V_t=0}$ the potential $V(x)$ is $G^{\mathrm{equiv}}_{V_t=0}$-equivalent to one from cases of Table 2.*

Table 2. Results of classification for the subclass $\{V = V(x)\}$. Here $\nu, \alpha, \beta \in \mathbb{R}$, $(\alpha, \beta) \neq (0,0)$.

N	N_1	V	mod G^{equiv}	Basis of $A^{\max}$
0	6	$V(x)$		$M,\ \partial_t$
1	7	$(\alpha+i\beta)x^{-2}$	$\beta \geq 0$	$M,\ \partial_t,\ D(t)$
2	7	$x^2+i+(\alpha+i\beta)x^{-2}$		$M,\ \partial_t,\ D(e^{4t})$
3	4	i		$M,\ \partial_t,\ \partial_x,\ G(t)$
4	4	$x+i\nu$	$\nu > 0$	$M,\ \partial_t,\ G(1)+tM,\ G(2t)+t^2M$
5	2	$-x^2+i\nu$	$\nu \geq 0$	$M,\ \partial_t,\ G(\sin 2t),\ G(\cos 2t)$
6	3	$x^2+i\nu,\ \nu \neq \pm 1$	$\nu \geq 0$	$M,\ \partial_t,\ G(e^{2t}),\ G(e^{-2t})$
7	5	0		$M,\ \partial_t,\ \partial_x,\ G(t),\ D(t)$
8	5	x		$M,\ \partial_t,\ G(1)+tM,\ G(2t)+t^2M,\ D(2t)+G(3t^2)+t^3M$
9	5	x^2+i		$M,\ \partial_t,\ G(e^{2t}),\ G(e^{-2t}),\ D(e^{4t})$

Proof. 2. Let $V = V(x)$ and $A^{\max}(V) \neq A^{\ker}_{V_t=0}$. Consider an arbitrary operator $Q = D(\xi) + G(\chi) + \lambda M \in A^{\max}(V)$. Under Lemma's assumption, the condition (3) implies a set of equations on V of the general form

$$(ax+b)V_x + 2aV = c_2x^2 + c_1x + \tilde c_0 + ic_0, \qquad \text{where} \qquad a, b, c_2, c_1, \tilde c_0, c_0 = \mathrm{const} \in \mathbb{R}.$$

The exact number k of such equations with the linear independent sets of coefficients can be equal to either 1 or 2. (The value $k = 0$ corresponds to the general case $V_t = 0$ without any extensions of $A^{\max}$.)

For $k = 1$ $(a, b) \neq (0,0)$ and there exist two possibilities $a = 0$ and $a \neq 0$. If $a = 0$ without loss of generality we can put $b = 1$. Condition (3) results in $\xi_t = 0$, $c_2 = c_0 = 0$, i.e. $V_x = c_1x + \tilde c_0$, and then $k = 2$ that is impossible.

Therefore, $a \neq 0$ and we can put $a = 1$. $\tilde c_0, b = 0 \bmod G^{\mathrm{equiv}}_{V_t=0}$. Condition (3) results in $\chi = 0$ (then $c_1 = 0$), $\lambda_t = 0$, $\xi_{tt} = 2c_0\xi_t$ and $c_2 = c_0^2$. Depending on $c_0 = 0$ and $c_0 \neq 0$, we obtain Cases 1 and 2 (of Table 2) correspondingly.

The condition $k = 2$ involves $V = d_2x^2 + d_1x + \tilde d_0 + id_0$. $\tilde d_0 = 0 \bmod G^{\mathrm{equiv}}_{V_t=0}$. Considering different possibilities for values of the constants d_2, d_1 and d_0, we obtain Cases 3–9:

$$d_2 = d_1 = d_0 = 0 \ \rightarrow\ 7; \qquad d_2 = d_1 = 0,\ d_0 \neq 0 \ \rightarrow\ 3;$$
$$d_2 = d_0 = 0,\ d_1 \neq 0 \ \rightarrow\ 8; \qquad d_2 = 0,\ d_0, d_1 \neq 0 \ \rightarrow\ 4;$$
$$d_2 < 0 \ \rightarrow\ 5; \qquad d_2 > 0,\ d_2 \neq d_0^2 \ \rightarrow\ 6; \qquad d_2 > 0,\ d_2 = d_0^2 \ \rightarrow\ 9.$$

Note 3. To prove Theorem 5, it is sufficient to consider only the case $k=1$, $a\neq 0$ in Lemma 3 since the other cases of extensions of $A^{\max}(V)$ with $V=V(x)$ admit operators of the form $G(\chi)+\lambda M$ ($\chi\neq 0$) and, therefore (by Corollary 1), are equivalent to Cases 1–5 of Table 1.

Note 4. The number N_1 for each line of Table 2 is equal to the number of the same or equivalent case in Table 1. The corresponding equivalence transformations have the form (4) where the functions T, X and Ψ are as follows:

$$2\to 7,\ 9\to 5:\quad T=-e^{-4t},\ X=\Psi=0;$$
$$6\to 3(\tilde\nu=\tfrac{1-\nu}{4}):\quad T=-e^{-4t},\ X=\Psi=0;\qquad 5\to 2(\tilde\nu=\nu):\quad T=\tan 2t,\ X=\Psi=0;$$
$$8\to 5:\quad T=t,\ X=-t^2,\ \Psi=\frac{t^3}{3};\qquad 4\to 4:\quad T=|\nu|t,\ X=-\sqrt{|\nu|}\,t^2,\ \Psi=\frac{t^3}{3}.$$

The results of the group classification obtained in this paper can be used to construct both invariant and partially invariant exact solutions of equations having the form (1). Moreover, we plan to study conditional symmetries of (1) to find non-Lie exact solutions.

Another direction for our future research to develop the above results is investigation of a more general class of $(1+n)$-dimensional NSchEs with potentials

$$i\psi_t+\Delta\psi+F(\psi,\psi^*)+V(t,\vec x)\psi=0, \tag{5}$$

where the $F=F(\psi,\psi^*)$ is an arbitrary complex-valued smooth function of the variables ψ and ψ^*. We have already described all possible inequivalent forms of the parameter-function F (without any restriction on the dimension n) for which an equation of the form (5) with a some potential V has an extension of the maximal Lie invariance algebra. We believe that the classification method suggested in this paper can be effectively applied to complete the group classification in (5) for the small values of n.

Acknowledgements

The authors are grateful to Profs. V. Boyko, A. Nikitin, I. Yehorchenko and A. Zhalij for useful discussions and interesting comments. The research of NMI was partially supported by National Academy of Science of Ukraine in the form of the grant for young scientists. ROP is grateful to Prof. F. Ardalan (School of Physics, Institute for Studies in Theoretical Physics and Mathematics, Tehran) for hospitality and support during writing this paper.

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